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ON THE RECONSTRUCTION OF SEPARABLE GRAPHS FROM ELEMENTARY CONTRACTIONS

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Some classes of separable graphs are reconstructed from the collection of elementary contractions. The results closely resemble those of the authors in the case of reconstruction from point-deleted subgraphs.

1. Introduction

Of the many variations of the Reconstruction Conjecture, the following three are quite recent.

"Any connected graph G with at least 5 (or 6 or 4) points can be uniquely reconstructed from all the elementary contractions (or elementary homomorphisms or elementary partitions, respectively)."

An elementary partition is an elementary contraction or an elementary homomorphism. These conjectures were proposed and the validity of the corresponding conjectures for disconnected graphs except for certain cases were established by Sampathkumar and Bhave [1]. Bhave, Kundu and Sampathkumar [2] have verified their validity for the case of trees. Kundu [6] has reconstructed unicyclic graphs from elementary contractions. A counter-example to the reconstruction from elementary contractions is given by the pair $K_{1,3}$ and P_4.

Henceforth we use the abbreviation e.c. for elementary contraction. In this paper we reconstruct from e.c.s a class of separable graphs which include those without endpoints. The procedure and results resemble those obtained by us [5] in the case of the Reconstruction from point deleted subgraphs.

In this paper we consider only connected simp[illegible]phs with at least five points and which are not trees. First, we give a few de[illegible]s and concepts which are explained by Fig. 1. Terms not defined he[illegible] ma[illegible]ound in Harary [4].

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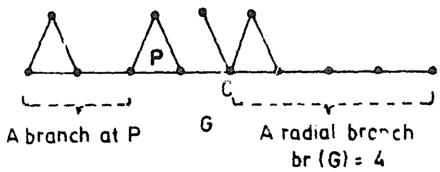

Fig. 1.

The block or cutpoint of G corresponding to the unique central point of the block-cutpoint-tree of G (denoted by bc(G)) is the *autocenter* of G, denoted by $C(G)$ or C. (The autocenter of the pruned graph is defined as the pruned center, by Greenwall and Hemminger [3]). If v is a cutpoint of G, then a maximal connected subgraph with v as a noncutpoint is a *branch at* v. If X is a block and v, a point of X, is a cutpoint of G then the union of all the branches at v except the one containing X is the *branch of* G *at* X, *rooted at* v. Branches are always rooted and a branch isomorphic to B is called a B-*branch* and its root a B-*root*.

The subgraph of G corresponding to a path of bc(G) is called a *block-path* in G. Let B be a block and let X be a block or cutpoint. The *block-distance* $D(X, B)$ is the number of blocks in the block-path (X, B) including B and excluding X, if X is a block. We define $D(B, B)=0$. $D(X, B)$ is k if the distance between the corresponding points of bc(G) is $2k$ (when X is a block) or $2k-1$ (when X is a cutpoint). *The block-length* of a block-path is the number of blocks in it. The block-length of a branch at X is $\max D(X, B)$ where B is any block in that branch.

A branch at C with maximum block-length, say br(G), is a *radial branch* and br(G) is called the *block-radius* of G. A block-path of maximum block-length is a *diametral block-path* and its block-length bd(G) $(=2\,\mathrm{br}(G)$ or $2\,\mathrm{br}(G)+1)$ is the *block-diameter of G*.

By a branch we always mean a branch at $C(G)$ unless otherwise specified.

By 'the branch is a path' we always mean a branch rooted at an endpoint of this path. Let $B(G)$ be the collection of blocks of G and $n(G)$ be the number of blocks in G.

Let $G(u, v)$ denote the e.c. of G obtained by contracting the line (u, v).

Let $\mathrm{EC}=\{G(u, v) \mid (u, v) \in E(G)\}$ be the family of e.c.s of G, multiplicities being taken into account.

2. Preliminary results

First we prove a few lemmas.

Lemma 1. *Let G be a block with at least three points and $u \in V(G)$. Then there are at least two distinct vertices $v_1, v_2 \in V(G)$ such that $G(u, v_1)$ and $G(u, v_2)$ are blocks.*

Proof. If possible, let G be such that G is the smallest block with a point u where at most one $G(u, v_i)$ is a block. Let $G(u, v)$ be a separable graph with k ($k \geq 2$) blocks. Then G is the union of k blocks, say $B_1, B_2, \ldots, B_k$ where $B_i \cap B_j$ ($i \neq j$) is the line (u, v). Since $|V(B_i)| < |V(G)|$, each B_i has a v_i ($\neq v$) such that $B(u, v_i)$ is a block. But each $G(u, v_i)$ is also a block, a contradiction which establishes the lemma. □

Corollary 1.1. *If G is a block, then there exist at least $|V(G)|$ e.c.s of G which are blocks.*

Since at most one e.c. of a separable graph is a block, EC determines whether G is a block or not.

Henceforth let G be a separable graph.

Lemma 2. *The collection $B(G)$ can be determined from* EC.

Proof. Of all the blocks occurring in all the e.c.s let B be one with maximum number of points. As this B could not have been obtained from a bigger block by contraction, B is an element of $B(G)$. Let D, a block, be an e.c. of B. Consider an H in EC with minimum number (possibly zero) of blocks isomorphic to B and with maximum number of blocks isomorphic to D. As in H one D must have been obtained by an e.c. of B, $B(G)$ is obtained from $B(H)$ by deleting one D and adding one B. □

Lemma 3. *The block-radius, the block-diameter and the auto-center can be determined from* EC.

Proof. Let H be a block-path which is not a tree and u and v be points of H such that $n(H(u, v)) = n(H)$. If $\mathrm{bd}(H)$ is k, then $\mathrm{bd}(H(u, v))$ is $k-1$ or k according as u and v are both cutpoints of H or not. (Note that the block-diameter can increase by at most one; this occurs only when $n(H(u, v)) > n(H)$.) Now by Lemma 1, there is at least one e.c. of H with block-diameter k. This observation leads to the following: Any $G(u, v) = H$ with $n(H) = n(G)$ and with maximum block-diameter, determines $\mathrm{br}(G)$, $\mathrm{bd}(G)$ and whether $C(G)$ is a cutpoint or a block.

Let C be a block. If G has at least two non-K_2 blocks, then, of the e.c.s just considered above, any one, say H, with maximum number of points in $C(H)$ gives $C(G)$, as the contraction must have taken place in a non-K_2 block different from $C(G)$. If G has only one non-K_2 block, say B, then consider

$$s = \max\{D(E, B) \mid E \text{ is an end block in any e.c. } H \text{ of } G \text{ with } n(H) = n(G) - 1\}$$

If $s = \mathrm{br}(G)$, then $C(G) = B$ and if $s \neq \mathrm{br}(G)$, then $C(G) = K_2$. □

Now, the following collections can be recognised in EC.

$$\mathrm{EC1} = \{H = G(u, v) \mid n(H) = n(G), C(H) \cong C(G)\},$$

$$\mathrm{EC2} = \{H = G(u, v) \mid n(H) = n(G), \mathrm{br}(H) = \mathrm{br}(G) \text{ and } C(H) \text{ is an e.c. of } C\},$$

$$\mathrm{EC3} = \{H = G(u, v) \mid C(H) \cong C(G)\}.$$

Note that for any e.c. in EC1 or EC3, $\mathrm{br}(H) = \mathrm{br}(G)$.

Lemma 4. *If C is a block, the branches of G (at C) can be determined from* EC.

Proof. We consider three cases according as $|V(C)|$ is 2, 3 or at least 4.

Case (i): $|V(C)| \geq 4$. In this case, $\mathrm{EC3} = \emptyset$ iff G has exactly two branches and they are paths (rooted at endpoints—according to our notation). So, let $\mathrm{EC3} \neq \emptyset$. Since $|V(C)| \geq 4$, $\mathrm{EC2} \neq \emptyset$. G has a trivial branch K_1 (at C), iff there is an H in EC2 with a trivial branch (at $C(H)$). If such an H exists, then any $G(u, v)$ in EC2 with maximum number of branches gives all the branches of G, since either at u or v, G must have a trivial branch.

Suppose all the branches are nontrivial. In this case, among the graphs in EC2, consider one with a branch having maximum number of points. This branch being the union of two branches of G, the others in this e.c. are branches of G.

Suppose there exists a known branch B and adjacent points u, v in B such that when B is replaced by $B\ (u, v)$ in G, the autocenter and block-radius are not altered; that is $C(G(u, v))$ is the same as $C(G)$ and $\mathrm{br}(G(u, v)) = \mathrm{br}(G)$. Such a $G(u, v)$ is in EC3.

Now consider all the graphs in EC3 with a $B(u, v)$-branch. Among these choose only those with a minimum number (possibly zero) of B-branches. Of these select one with a maximum number of $B(u, v)$-branches. In this graph clearly one B-branch must have been contracted into a $B(u, v)$-branch and hence all the branches of G can be determined.

Our aim now is to find such a known branch B. It is easy to verify that the following choices satisfy our requirements.

If there is a known non-radial branch B, then any $B(u, v)$ will serve our purpose. Suppose all the known branches are radial. If one such branch B is not a path, it is possible to choose points u and v in B such that $B(u, v)$ does not alter the block-radius of G. Suppose all the known branches are paths. If $|V(C)| > 4$, then there are at least three known branches and in this case choose any known branch.

Suppose $|V(C)| = 4$. If the union of the two unknown branches is a radial branch, choose any known branch as B. Suppose both the unknown branches are non-radial. Here, $C(G(u, v))$ is a cutpoint, exactly one radial branch is a path and $\mathrm{br}(G(u, v)) = \mathrm{br}(G)$ iff both u and v are in one of these radial branches. In such a $G(u, v)$, the block $C(G)$ can be recognised (as the only non-K_2 block containing $C(G(u, v))$ and from this the branches of G can be easily determined.

Case (ii): $C = K_3$. Here the number of e.c.s in EC2 is the same as the number of radial branches of G.

If all the three branches are radial, of all the radial branches in the graphs in EC3, one with maximum number of points is a branch of G. Using this as B above, the other branches can be determined.

Suppose only two branches are radial. Now both the graphs in EC2 are paths iff the two radial branches of G are paths and the other branch is trivial. In both the graphs of EC2, exactly one branch is a path iff both the radial branches are paths and the other branch is nontrivial, which is determined from any of these graphs. Hence, let at least one radial branch be different from a path. Here EC3 $\neq \emptyset$ and the maximum number of branches in any graph in EC3 gives the number of branches in G. The third branch is trivial iff both the graphs in EC2 are isomorphic and they give the branches of G. If the third is nontrivial, an e.c. in EC3 with maximum number of points in its nonradial branch gives the nonradial branch of G. Using this as B, as before, the other branches are determined.

Case (iii): $C = K_2$. Suppose there is an H in EC3 with both branches different from paths. Then both branches of G are different from paths and of all the branches of graphs in EC3, one with maximum number of points is a branch of G and using this the other branch is determined.

Suppose in a graph H of EC3, both the branches are paths. This is possible iff G is a path except that one K_2 is replaced by a K_3. By finding the distance of this K_3 from the endblocks, the branches of G are determined.

If all the graphs in EC3 have exactly one branch as a path, then so is G or one branch of G is a path except for one K_3 instead of a K_2, and the other branch is similar to this or a path of length br(G) but for one 'extra K_2' (see Fig. 2a and 2b).

If it is known that one branch of G is a path, then choose an e.c. H such that $n(H) = n(G) - 1$, $\text{br}(H) = \text{br}(G)$, $C(H)$ is a cutpoint and a point of degree at least 3 is nearest to $C(H)$. Since such an H is isomorphic to a $G(u, v)$ where u and v

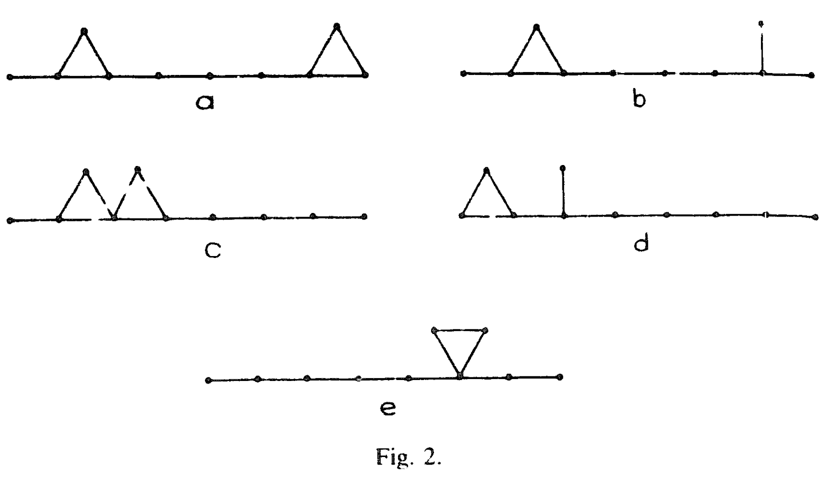

Fig. 2.

are in the branch which is a path, branches of G are the same as branches of H at $C(H)$. So, now we have to ascertain whether a path is a branch of G or not.

The conditions $n(G)=2\,\mathrm{br}(G)+1$, exactly two blocks of G are K_3's and the remaining are K_2's give all graphs similar to those in Figs. 2a and 2c. The conditions $n(G)=2\,\mathrm{br}(G)+2$ and all but one K_3 are K_2's in G give graphs similar to those in Figs. 2b, 2d and 2e. In the former case the block-distances of the K_3's from the endblocks and also from each other are determined from EC and hence the branches determined (the details are omitted). In the latter case, a graph in which K_3 intersects an end K_2 comes under both the types given by Figs. 2d and 2e. For definiteness let us include it in type 2d (of graphs given by Fig. 2d). Now graphs of type 2b and 2d are distinguished from those of type 2c, by the presence of an e.c. H with $C(H)=K_2$, $\mathrm{br}(H)=r$ $(=\mathrm{br}(G))$ and having a K_3 in a diametral block path. This H also indicates whether the K_3 is an endblock or not in type 2b and 2d.

Fig. 3.

Suppose this K_3 is an end block of G. If $r=1$, the graphs are as shown in Fig. 3, and they are distinguished from their ECs. So, let $r\geqslant 2$. Consider an e.c. H such that $B(H)=B(G)-\{K_2\}$, $C(H)$ is a cutpoint, H has just two endpoints and exactly two branches, both of which are different from paths. If no such H exists then one branch of G is a path; otherwise both the branches are different from paths and from an e.c. in which the K_3 is contracted to a K_2, the branches of G are determined.

Suppose K_3 is not an end block of G. Consider an e.c. H with four endpoints (Note that $C(H)$ is a cutpoint). Both the branches of G are different from paths iff (a) both the branches of H are different from paths or (b) H has three branches and one is a K_2, the second is a path and the third is different from a path. In case (a), the branch with an 'extra' K_2 is determined from any e.c. in EC1 and the other branch determined as usual. In case (b) in the H just described above, replacing the 'extra' K_2 in the branch which is not a path by a K_3, G is reconstructed.

This completes the proof. □

Lemma 5. *If C is a cutpoint, then whether the number of branches of G is two or more can be determined from EC.*

Proof. If $n(G)=2$ o 3, then G has exactly 2 or 3 branches respectively. So, let $n(G)\geqslant 4$. If all the graphs in EC3 have at least 3 branches, then so has G.

Suppose there is one e.c. in EC3 with only two branches. If G has three branches then one is a K_2 and hence there is only one such e.c. in EC3. If G has only two branches, then at least two e.c.s in EC3 have exactly two branches. This determines wether the number of branches is two or more. □

Lemma 6. *If C is a cutpoint, then the branches of G at C or at a block D containing C can be determined from* EC.

Proof. *Case* 1: G has at least 3 branches.

First find a branch with maximum number of points from EC3. If this branch is not a path, using this branch, the other branches are determined as usual. Otherwise, this is a radial branch and all the radial branches are paths. Now, in an e.c. where the contraction is in one of the radial branches, $C(G)$ is located and the other branches are determined.

Case 2: G has only 2 branches.

Determination of whether one branch is a path or not and construction of the branches in case no branch is a path is similar to those in the case when C is a K_2 and is hence omitted.

So, let G have exactly two branches and let one of them be a path. In $\{C(H) \mid H \in EC,\ n(H) = n(G) - 1\}$, the one, say D, with maximum number of points is a block containing C. If D is also a K_2, then considering an e.c. say H, such that $C(H) = K_2$ and the block distance of $C(H)$ to a block of H which is not a K_2, is minimum, we can reconstruct G; for H is isomorphic to an e.c. of G obtained by contracting one K_2 in the radial branch which is a path. If D is not a K_2, consider an e.c., say H, whose autocenter is D. In H consider the branches of D (if $n(G) = 2$, then one K_2 is the only nontrivial branch at D. So, let $n(G) \geq 3$). If there is only one branch which is a path of maximum length, say P_k, then by replacing this path by P_{k+1} we can reconstruct G. In the case where there are at least two branches of H which are paths of maximum length, we can find the branches of G at D, by replacing one branch of D (in H) isomorphic to P_k by P_{k+1}. □

Note that if C is a cutpoint and if all the branches are known, then G is reconstructed by identifying all the roots of the branches.

3. Reconstruction

Theorem 7. *Let C be a block. If there exists a branch B of G such that one of its e.c.s is not a branch of G, then G can be reconstructed from* EC.

Proof. Let $B(u, v)$ be not a branch of G. Let the block length of B be n. Then the block length of $B(u, v)$ is n or $n-1$ or $n+1$.

Case 1: Block length of $B\ (u, v)$ is n.

Consider an e.c. in EC3 which has minimum number of branches (possibly zero) isomorphic to B, one branch isomorphic to $B(u, v)$ and the other branches same as those of G. By replacing the branch $B(u, v)$ by B, G is reconstructed.

Case 2: Block length of $B(u, v)$ is $n-1$. If B is not a radial branch or B is radial and there exist at least three radial branches, then proceed as in Case 1.

Suppose B is one of the only two radial branches. If one of the radial branches has at least $m+3$ points or both radial branches have $m+2$ points (where $\mathrm{br}(G)=m$), then we can easily see that one of the radial branches satisfies the condition for Case 1. So, let the number of points in the radial branches be $m+1$ and $m+2$ or both $m+1$. Now, consider the e.c.s with the autocenter as a cutpoint. In all these e.c.s we can easily find C and the branches at this block. From the e.c. in which one $B-u$ appears instead of a B, G is reconstructed by replacing the branch $B(u, v)$ by B.

Case 3: Block length of $B(u, v)$ is $n+1$.

If $n+1 \leq m$, then proceed as in Case 1. Suppose $n=m$. Let k be the maximum number of points in any branch of G, except B. If $|V(B)| \geq k+2$, then the condition for case 1 will be satisfied. So let $|V(B)| \leq k+1$. Consider an e.c. H, such that $C(H)$ is a cutpoint, $\mathrm{br}(H)=\mathrm{br}(G)+1$, and a block isomorphic to C containing the point $C(H)$ and all the branches of G except one B which is replaced by a $B(u, v)$ and the branch $B(u, v)$ is at the point $C(H)$ of C. (Note that the choice for $C(G)$ in H is unique since $|V(B)| \leq k+1$). G is reconstructed by replacing this $B(u, v)$ by B. □

Corollary 7.1. *If G is a separable graph without endpoints, then G is reconstructable from* EC.

Corollary 7.2. *If C is a block and there exists a branch B of G such that $|V(B)| \geq |V(C)|$, then G is reconstructable from* EC.

Theorem 8. *Let C be a block. If there exists a branch B of G, such that an e.c. of B is not a branch of C at any point similar to a B-root in C, then G can be reconstructed from* EC.

This theorem and its proof are similar to those in the case of the reconstruction from point deleted subgraphs, given in [5].

Suppose C is a cutpoint and only the branches at D are known. Then we can prove results similar to Theorems 7 and 8 where C is replaced by D.

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